Int. J. Solids Structures Vol. 31, No. 1, pp. 109–126, 1994
Printed in Great Britain

0020–7683/94 $6.00+ .00

GENERALIZED CHARACTERISTIC METHOD OF ELASTODYNAMICS

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(*Received* 17 *August* 1992; *in revised form* 9 *July* 1993)

Abstract—A generalized characteristic method to solve two-dimensional time domain elastodynamics problems is developed. The (real) characteristic surfaces in x, y, t-space for the Navier's equations are continued to the complex domain, in which they are called generalized characteristics. The first order ordinary differential equations along the generalized characteristics which are equivalent to the Navier's equations are used to simplify an initial-boundary value problem of elastodynamics to a set of algebraic equations whose unknowns are the first order derivatives of the displacement components. The reflection coefficients of longitudinal and transverse waves, whose fronts can be of arbitrary convex shape, from a traction free boundary are also found to be algebraic expressions. The method makes the inversion of Laplace transform required by Cagniard's technique unnecessary and can be used directly to analyse the head wave and the regions of influence of the reflected waves. Closed form expressions for the time domain Green's function in the whole plane and the solution of Lamb's problem for a buried concentrated force are given as applications of the method. Formulae for the surface displacement take on a simpler form than those reported previously. Finally, some numerical results for the displacement of receivers on and below the surface are given. The displacements of reflected waves approach infinity when they are treated separately but superposing the displacements due to the reflected p- and s-waves shows that the transient Rayleigh pulse is finite.

1. INTRODUCTION

The method of characteristics has been used successfully to treat one-dimensional elastic wave propagation problems (Achenbach, 1973; Eringen and Suhubi, 1975; Miklowitz, 1978). However, this technique has not been used efficiently to address elastic wave propagation problems in two or three dimensions [see Clifton (1967) and Ziv (1969)]. That is, there is no solution of the wave equation in two dimensions which is comparable with d'Alembert's solution in one dimension (Achenbach, 1973; Miklowitz, 1978; Hudson, 1980). The difficulty in using the characteristic method in two-dimensional elastodynamics is due to the fact that the governing equations are not hyperbolic in some regions of x, y, t-space where waves propagate. For the special case of self-similar problems, the difficulty has been circumvented [see Craggs (1960) and Miles (1960)]. However, the characteristic method has not been applied successfully to the general two-dimensional problem.

An example problem that has not been solved with the characteristic method is that of the motion of an elastic half-space loaded by a buried impulsive line or point force first investigated by Lamb (1904). In the papers by Pekeris (1940, 1955), Pekeris and Lifson (1957), Garvin (1954), Pao and Gajewski (1977), Tsai and Ma (1991) and in the books by Ewing *et al.* (1957), Achenbach (1973), Eringen and Suhubi (1975), Miklowitz (1978) and Hudson (1980), this problem was studied using double integral transforms. It should be mentioned that Pekeris (1940, 1955) inverted the transformed solution for the displacements by a technique close in nature to Cagniard's (1962) and illustrated his solution with numerical examples. Hudson (1980) was the first to calculate closed form expressions for displacements in a half-plane subject to a buried line impulse. The transient displacement field in an elastic half space induced by surface impulse loading was studied originally by

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Lamb (1904) and has been studied by many authors using integral transform methods (Payton, 1967; Miklowitz, 1978), Chaplygin's transformations (Craggs, 1960), and Smirnov-Sobolev's method (Smirnov and Sobolev, 1932; Thompson and Robinson, 1969; Cherepanov and Afanas'ev, 1974). The last two methods can be applied only to self-similar problems. Additional references concerning the problems can be found in the excellent review papers of Miklowitz (1964) and Pao (1983).

A generalized characteristic method was presented previously (Su and Liu, 1986; Su and Fu, 1989). In the present paper the method is developed and completed. It can be thought of as the two-dimensional development of the characteristic method analogous to d'Alembert's solution for one-dimensional problems. In Section 2, the classical characteristic surfaces and the equivalent first order ordinary differential equations of the Navier equations along the characteristics are recalled. In Section 3 the real characteristics are continued into the complex plane in which they are called generalized characteristics. The generalized characteristics have the same properties as classical characteristics so that they can be used to solve two-dimensional elastodynamic problems in the complete x, y, t-space. From the expressions for stress components along the generalized characteristics the initial-boundary value problems of elastodynamics are simplified to a set of algebraic equations whose solutions are first order derivatives of the displacement components. The method leads directly to the reflection coefficients of elastic waves, whose fronts can be of arbitrary convex shape, from a traction free boundary along the reflected characteristics such as those found for the reflection of plane waves (Section 4). The full plane Green's function is found as the solution of a boundary value problem (Section 5). The generalized characteristics are used directly to analyse the regions of reflected waves and head waves. The reflection coefficients and the full plane Green's function lead to the solution of Lamb's problem for a buried source. This solution yields particular compact solutions for the surface displacement and for subsurface displacements induced by a surface force. Some numerical results for the displacement histories of receivers near and far from the source in the half plane are shown. The results show the salient features of the displacements of these receivers. In particular it is shown that the displacements of the reflected waves approach infinity when they are treated separately but that summing the displacements of the reflected p- and s-waves reveals that the transient Rayleigh pulse is finite.

2. CHARACTERISTICS FOR NAVIER'S EQUATIONS

Navier's equation for the displacement vector $\mathbf{u}$ for an elastic half space $(-\infty < x < \infty, y \geqslant 0)$ subjected to a buried impulsive line force (plane strain) is written as:

$$(\lambda+2\mu)\nabla\cdot\nabla\mathbf{u}+(\lambda+\mu)\nabla\times\nabla\times\mathbf{u}-\rho\frac{\partial^2\mathbf{u}}{\partial t^2} = -\rho\mathbf{I}\delta(x-x_0)\delta(y-y_0)\delta(t), \tag{1}$$

where λ and μ are Lamé's elastic constants, ρ is mass density, $\mathbf{I}$ is the load vector, δ is the Dirac delta function, and x_0 and y_0 are the coordinates of the point of load application. Hooke's Law relates the stress components to the displacement gradient through

$$\sigma = \lambda\nabla\mathbf{u}[\mathbf{1}]+\mu(\nabla\mathbf{u}+\mathbf{u}\nabla) \tag{2}$$

and the zero initial conditions are

$$\mathbf{u} = \frac{\partial\mathbf{u}}{\partial t} = 0 \quad \text{for} \quad t = 0. \tag{3}$$

According to the theory of differential equations (Courant and Hilbert, 1962), surfaces $z(x, y, t) = C$ (C is a real number) with $\nabla z(x, y, t) \neq 0$ in x, y, t-space are said to be characteristics of eqn (1) if they satisfy the characteristic condition which can be simplified to

$$\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2-a_1^2\left(\frac{\partial z}{\partial t}\right)^2\right)\left(\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2-a_2^2\left(\frac{\partial z}{\partial t}\right)^2\right)=0, \tag{4}$$

where a_1 and a_2 are the slownesses of longitudinal and transverse waves respectively. Appendix A shows that the implicit functions $z_1(x, y, t) = C_1$ and $z_2(x, y, t) = C_2$ determined by

$$t-z_i x\pm\eta_i(z_i)y+\phi_i(z_i)=0, \quad \eta_i(z)=\sqrt{a_i^2-z^2}, \quad i=1,2 \tag{5}$$

are the general integrals of eqn (4). Here $-a_i \leqslant C_i \leqslant a_i$ $(i = 1, 2)$, z_i are real functions, and $\phi_1(z_1)$ and $\phi_2(z_2)$ are arbitrary functions whose second derivatives exist. Substitution of $z_i(x, y, t) = C_i$ into eqns (5) shows that they define two couples of families of characteristic planes in x, y, t-space. The $+$ and $-$ signs in front of the radicals indicate the directions of wave travel. The functions $\phi_i(z_i)$ can be determined for a special point in x, y, t-space through which the characteristic plane passes.

If a family of the characteristic planes on a parameter possesses an envelope, then this envelope is also a solution of eqn (4). The characteristic planes [eqns (5)] are the planes tangent to the envelope. The shape of the envelope depends on the functions $\phi_i(z_i)$. Eliminating z_i from the following equations [see Sneddon (1957) and Courant and Hilbert (1962)]:

$$\begin{aligned} t-xz_i\pm\eta_i(z_i)y+\phi_i(z_i)&=0,\\ -x\mp yz_i/\eta_i(z_i)+\mathrm{d}\phi_i(z_i)/\mathrm{d}z_i&=0,\end{aligned} \tag{6}$$

leads to implicit equations of envelopes of characteristic planes of the form

$$f_i(x, y, t) = 0. \tag{7}$$

The envelopes are called the singular integrals of eqn (4). Equations (6) were also given as saddle points of the paths of the transform inversion by Cagniard's technique [see Norwood (1975)].

For the special case $\phi_i(z_i) \equiv 0$, the envelopes are the characteristic cones

$$t-a_i\sqrt{x^2+y^2}=0, \quad i=1,2. \tag{8}$$

The characteristic planes and their envelopes can be wave fronts which define the regions of influence of waves [see Courant and Hilbert (1962)].

In Section 5 the problem of inhomogeneous field equations and homogeneous boundary conditions will be transformed to one of homogeneous field equations and inhomogeneous boundary conditions, therefore it is sufficient to consider the homogeneous field equations here. With the aid of interior derivatives [see Courant and Hilbert (1962)], the relationships along the characteristics furnished by the differential equations can be written in a simple manner. Write the displacement components of the vector $\mathbf{u}$ as $U_i(z_i)$, $V_i(z_i)$ $i = 1, 2$. Substituting eqns (5) into eqn (1) leads to the following differential equations along characteristic planes:

$$\begin{aligned} \eta_1(z_1)G_1(z_1) &= \mp z_1 H_1(z_1), \quad z_2 G_2(z_2) = \pm\eta_2(z_2)H_2(z_2),\\ G_i(z_i) &= \mathrm{d}U_i(z_i)/\mathrm{d}z_i, \qquad H_i(z_i) = \mathrm{d}V_i(z_i)/\mathrm{d}z_i. \end{aligned} \tag{9}$$

Here the upper and lower signs depend on the signs of the radicals in eqn (5). Equations (9) are called the interior operators of Navier's equation along the characteristic surfaces.

3. CONTINUATION TO THE COMPLEX DOMAIN

Equation (1) is hyperbolic when the point (x, y, t) is located outside the envelope of longitudinal wave characteristics, mixed when the point is located between envelopes of the longitudinal and transverse waves, and elliptic when the point is located within the envelope of transverse waves [see Courant and Hilbert (1962) and for a special case see Miles (1960) and Craggs (1960)]. That is, the classifications of eqn (1) are based solely on position in x, y, t-space.

Lewy [see Courant and Hilbert (1962)] discussed the transition in type from elliptic to hyperbolic for second order differential equations (two independent variables) by considering the complex domain. The continuation to the complex plane makes the distinction between equation classifications disappear in principle. In the same way, the real value of $z_i(x, y, t) = C_i(-a_i \leqslant C_i \leqslant a_i)$ can also be continued to the whole complex plane. As C_i are not only real $(-a_i \leqslant C_i \leqslant a_i)$, but also complex, the implicit complex functions z_i (the same symbols are used for both the real and complex functions)

$$t - z_i x \pm \eta_i(z_i) y + \phi_i(z_i) = 0, \quad i = 1, 2, \tag{10}$$

also satisfy the characteristic condition [eqn (4)]. Here the branch cuts of the radical $\eta_i(z_i)$ are taken along $\mathrm{Im}\,[z_i] = 0, a_i < |\mathrm{Re}\,[z_i]| < \infty$ which are the branch cuts of the functions $\sqrt{a_i + z}$ and $\sqrt{a_i - z}$, respectively. The branch of the function η_i that satisfies eqn (10) is the only branch used in subsequent equations (Appendix B). The real and complex solutions of the characteristic condition are called "generalized characteristics". However, according to the classical theory, the complex solutions are not characteristic surfaces.

The solutions of eqn (1) can be expressed as the real part of complex functions (Courant and Hilbert, 1962) as

$$u(x, y, t) = \mathrm{Re}\,\{U_1(z_1) + U_2(z_2)\}, \quad v(x, y, t) = \mathrm{Re}\,\{V_1(z_1) + V_2(z_2)\}. \tag{11}$$

Since the solutions are analytic functions, the approach used in the previous section for the real characteristics can be pursued for the complex characteristics. The same equations are found for the complex functions along complex "characteristic surfaces" (Su and Liu, 1986; Su and Fu, 1989) so that

$$\eta_1(z_1) G_1(z_1) = \mp z_1 H_1(z_1), \quad z_2 G_2(z_2) = \pm \eta_2(z_2) H_2(z_2). \tag{12}$$

Equations (10) and (12) can be used to prove that

$$\partial U_1(z_1)/\partial y - \partial V_1(z_1)/\partial x = 0, \quad \partial U_2(z_2)/\partial x + \partial V_2(z_2)/\partial y = 0. \tag{13}$$

Equations (13) show that there is a longitudinal wave and transverse wave traveling (the speeds are the same as those traveling along the real characteristics) along the generalized characteristics z_1 and z_2, respectively. From eqn (2), the stress components on generalized characteristics are expressed as

$$\sigma_x = \mu\, \mathrm{Re}\left[\frac{\gamma(\eta_1(z_1))}{z_1^2} G_1(z_1) \frac{\mathrm{d}z_1}{\mathrm{d}x} + 2G_2(z_2) \frac{\mathrm{d}z_2}{\mathrm{d}x}\right],$$

$$\sigma_y = \mu\, \mathrm{Re}\left[\frac{\gamma(z_1)}{z_1^2} G_1(z_1) \frac{\mathrm{d}z_1}{\mathrm{d}x} - 2G_2(z_2) \frac{\mathrm{d}z_2}{\mathrm{d}x}\right], \quad \gamma(z) = a_2^2 - 2z^2,$$

$$\tau_{xy} = \mp \mu\, \mathrm{Re}\left[2 \frac{\eta_1(z_1)}{z_1} G_1(z_1) \frac{\mathrm{d}z_1}{\mathrm{d}x} + \frac{\gamma(z_2)}{z_2 \eta_2(z_2)} G_2(z_2) \frac{\mathrm{d}z_2}{\mathrm{d}x}\right], \tag{14}$$

where the sign for τ_{xy} is taken to be consistent with the sign of eqn (5).

The geometry of the complex functions is considered here. In the complex equations (10), z_i can be any value in the complex plane. Each expression of eqns (10) can be separated into two real equations which correspond to real planes in x, y, t-space. The solution of the real simultaneous equations corresponds to the intersection of the two planes. In other words, every complex function $z_i(x, y, t) = C_i$ corresponds to a straight line in x, y, t-space. All of these straight lines are on the side of the envelope opposite the characteristic planes [eqns (5)] so that each of eqns (10) defines the correspondence between the straight line and the complex value of C_i. There are two couples of families of the lines corresponding to the + or − signs in front of the radicals in eqns (10). However, when the wave propagates into the static region there is only one pair of generalized characteristics.

The generalized characteristic method is closely related to previous solution techniques. In particular, if $\phi_i(z_i) \equiv 0$, eqns (10) are Smirnov–Sobolev's transformation (Smirnov and Sobolev, 1932; Thompson and Robinson, 1969) as well as Chaplygin's transformation (Miles, 1960). It can also be shown that eqns (10) are the paths of integration in the Cagniard method used by de Hoop (1960).

4. REFLECTED WAVES AND THEIR CHARACTERISTICS

In this section, the generalized characteristic method is used to calculate free surface reflection coefficients. The boundary conditions on the traction free surface of the half plane ($-\infty < x < \infty, y \geqslant 0$) are given by

$$\sigma_y = 0, \quad \tau_{xy} = 0, \quad \text{on} \quad y = 0. \tag{15}$$

First the reflection of a longitudinal (p) wave is analysed. The generalized characteristic of the p-wave incident to the boundary is written as

$$t - z_1 x + \eta_1(z_1)y + \phi_1(z_1) = 0. \tag{16}$$

Reflected waves are generated when the incident wave impinges upon the boundary. The reflected waves propagate along characteristics which also satisfy the characteristic condition of eqn (4). According to eqns (5) and (10), there are two additional generalized characteristics which pass through the points on the boundary $x-t$ plane in x, y, t-space through which the incident waves pass. The y-component of their traveling direction is opposite to that of the incident wave and their characteristics are represented by

$$t - \xi_i x - \eta_i(\xi_i)y + \phi_i(\xi_i) = 0, \quad i = 1, 2, \tag{17}$$

where ξ_1 and ξ_2 are the characteristics of the reflected p- and s-waves, respectively.

The displacement components of the incident p-wave and the reflected p- and s-waves are given as

$$\begin{aligned} u_{pi}(x, y, t) &= \operatorname{Re}\{U_{pi}(z_1)\}, \quad u_{pr}(x, y, t) = \operatorname{Re}\{U_{pp}(\xi_1) + U_{ps}(\xi_2)\}, \\ v_{pi}(x, y, t) &= \operatorname{Re}\{V_{pi}(z_1)\}, \quad v_{pr}(x, y, t) = \operatorname{Re}\{V_{pp}(\xi_1) + V_{ps}(\xi_2)\}. \end{aligned} \tag{18}$$

On the boundary $y = 0$, both eqns (16) and (17) can be simplified to

$$t - \lambda x + \phi_1(\lambda) = 0, \tag{19}$$

where $\lambda = z_1(x, 0, t) = \xi_1(x, 0, t) = \xi_2(x, 0, t)$. The stress components for the incident and reflected waves can be calculated from eqns (10) and (14).

Substituting eqns (16)–(19) and eqns (14) into eqns (15) yields

$$\gamma(\lambda)G_{pp}(\lambda)-2\lambda^2 G_{ps}(\lambda) = -\gamma(\lambda)G_{pi}(\lambda),$$
$$2\eta_1(\lambda)\eta_2(\lambda)G_{pp}(\lambda)+\gamma(\lambda)G_{ps}(\lambda) = 2\eta_1(\lambda)\eta_2(\lambda)G_{pi}(\lambda). \tag{20}$$

Equations (20) with eqns (12) can be solved for the reflected wave components as

$$G_{pp}(\xi_1) = K^u_{pp}(\xi_1)G_{pi}(\xi_1), \quad G_{ps}(\xi_2) = K^u_{ps}(\xi_2)G_{pi}(\xi_2),$$
$$H_{pp}(\xi_1) = K^u_{pp}(\xi_1)H_{pi}(\xi_1), \quad H_{ps}(\xi_2) = K^v_{ps}(\xi_2)H_{pi}(\xi_2), \tag{21}$$

where

$$K^u_{pp}(\xi_1) = [4\xi_1^2\eta_1(\xi_1)\eta_2(\xi_1)-\gamma^2(\xi_1)]/R(\xi_1), \quad K^v_{pp}(\xi_1) = -K^u_{pp}(\xi_1),$$
$$K^u_{ps}(\xi_2) = [4\eta_1(\xi_2)\eta_2(\xi_2)\gamma(\xi_2)]/R(\xi_2), \quad K^v_{ps}(\xi_2) = [4\xi_2^2\gamma(\xi_2)]/R(\xi_2),$$
$$R(\xi_i) = 4\xi_i^2\eta_1(\xi_i)\eta_2(\xi_i)+\gamma^2(\xi_i). \tag{22}$$

The reflection of incident s-waves is handled in the same manner. The generalized characteristics of the incident and reflected waves are

$$t-xz_2+\eta_2(z_2)y+\phi_2(z_2) = 0,$$
$$t-x\zeta_i-\eta_i(\zeta_i)y+\phi_2(\zeta_i) = 0, \quad i = 1,2. \tag{23}$$

Again the first derivatives of the displacement components of reflected waves due to the incident s-wave are

$$G_{sp}(\zeta_1) = K^u_{sp}(\zeta_1)G_{si}(\zeta_1), \quad G_{ss}(\zeta_2) = K^u_{ss}(\zeta_2)G_{si}(\zeta_2),$$
$$H_{sp}(\zeta_1) = K^v_{sp}(\zeta_1)H_{si}(\zeta_1), \quad H_{ss}(\zeta_2) = K^v_{ss}(\zeta_2)H_{si}(\zeta_2), \tag{24}$$

where

$$K^u_{sp}(\zeta_1) = K^v_{ps}(\zeta_1), \quad K^v_{sp}(\zeta_1) = K^u_{ps}(\zeta_1),$$
$$K^u_{ss}(\zeta_2) = K^v_{pp}(\zeta_2), \quad K^v_{ss}(\zeta_2) = K^u_{pp}(\zeta_2). \tag{25}$$

The functions $K_{mn}(m, n = p, s)$ are the reflection coefficients of elastic waves along generalized characteristics. Knowing the functions $\phi_i(z_i)$ and the incident displacements, the reflected waves can be obtained using K_{mn}.

5. THE GREEN'S FUNCTION FOR THE WHOLE PLANE

The response of the unbounded medium subjected to an impulse [in eqn (1)] can be separated into two problems: the vertical load I_y solution is antisymmetric with respect to the x-axis and the horizontal load I_x solution is symmetric with respect to the x-axis. The symmetric problem can be reformulated as the following boundary value problem for a half plane:

$$\sigma_y^0 = \pm\tfrac{1}{2}\rho I_y\delta(x-x_0)H(t), \quad u^0 = 0 \quad \text{at} \quad y-y_0 = 0, \tag{26}$$

where

$$f^0 = \int_0^t (f)\,\mathrm{d}t, \tag{27}$$

the homogeneous form of eqn (1) is used, $H(t)$ is the Heaviside step function, and the $-$ sign is used for the half plane $y-y_0 \geqslant 0$ and the $+$ sign is used for the half plane $y-y_0 \leqslant 0$.

The generalized characteristics are expressed as

$$t-(x-x_0)z_i \pm \eta(z_i)(y-y_0) = 0, \quad i = 1,2 \tag{28}$$

and

$$\phi(z_i) = x_0 z_i \mp y_0 \eta_i(z_i) \tag{29}$$

since the generalized characteristics pass through the point $(x_0, y_0, 0)$ in x, y, t-space.

With the aid of eqns (14), the boundary conditions [eqn (26)] may be written as

$$\begin{aligned} t-\lambda(x-x_0) &= 0, \\ \gamma(\lambda)G_{12}(\lambda)-2\lambda^2 G_{22}(\lambda) &= \mp a_2^2 A_2 \lambda, \\ G_{12}(\lambda)+G_{22}(\lambda) &= 0, \end{aligned} \tag{30}$$

where the following expressions are used

$$t\delta(x-x_0) = \delta\left(\frac{x-x_0}{t}\right), \quad \delta\left(\frac{1}{\lambda}\right) = \pm \mathrm{Re}\left[\frac{\mathrm{i}\lambda}{\pi}\right], \quad A_2 = \pm \frac{\mathrm{i}I_y}{2\pi}. \tag{31}$$

Equations (31) are derived in Appendix C. The positive sign is used in the second and third expressions of eqns (31) if we choose $\mathrm{Im}\,(z_i) < 0$ while the negative sign is used if we choose $\mathrm{Im}\,(z_i) > 0$ (see Appendix C).

The solution is found as

$$\begin{aligned} G_{12}(z_1) &= \mp A_2 z_1, & G_{22}(z_2) &= \pm A_2 z_2, \\ H_{12}(z_1) &= A_2 \eta_1(z_1), & H_{22}(z_2) &= A_2 z_2^2/\eta_2(z_2). \end{aligned} \tag{32}$$

The response of the unbounded medium to the horizontal impulse is symmetric to the plane $y-y_0 = 0$ and the procedure of the previous paragraph leads to

$$\begin{aligned} G_{11} &= A_1 z_1^2/\eta_1(z_1), & G_{21} &= A_1 \eta_2(z_2), \\ H_{11} &= \mp A_1 z_1, & H_{21} &= A_1 z_2, \end{aligned} \tag{33}$$

where

$$A_1 = \frac{\mathrm{i}I_x}{2\pi}.$$

Equations (27), (28), (32) and (33) can be used to represent the Green's function for the whole plane as

$$\begin{aligned} u_j(x,y,t) &= \sum_{i=1}^{2} \frac{\mathrm{d}U_i^0}{\mathrm{d}z_i}\frac{\mathrm{d}z_i}{\mathrm{d}t} = \mathrm{Re}\sum_{i=1}^{2} G_{ij}(z_i)S_i(z_i), \\ v_j(x,y,t) &= \sum_{i=1}^{2} \frac{\mathrm{d}V_i^0}{\mathrm{d}z_i}\frac{\mathrm{d}z_i}{\mathrm{d}t} = \mathrm{Re}\sum_{i=1}^{2} H_{ij}(z_i)S_i(z_i), \quad j = 1,2, \end{aligned} \tag{34}$$

where

$$S_i(z_i) = [(x-x_0)-|y-y_0|z_i/\eta_i(z_i)]^{-1}.$$

The denominators in the above, $S_i(z_i)$, are zero at the wavefronts indicated by eqns (6) or (8). Thus $S_i(z_i)$ automatically gives the regions of influence of the waves.

6. THE SOLUTION OF LAMB'S PROBLEM FOR A BURIED FORCE

Before the elastic waves meet the boundary, the disturbance caused by the elastic waves in the half plane are the same as that for the whole plane. The solution of Lamb's problem for a buried force is found by superposing to the whole plane Green's function the contribution from the waves reflected from the boundary.

The generalized characteristics for the incident waves are eqns (28). Using eqns (17) and (23) in Section 4, the generalized characteristics for reflected waves are rewritten as

$$\begin{gathered} t-xz_i-\eta_i(z_i)y+x_0z_i-\beta_i(z_i)y_0=0, \quad \beta_i(z)=\sqrt{b_i^2-z^2}, \\ a_1=a_3=a_5=b_1=b_3=b_4, \quad a_2=a_4=a_6=b_2=b_5=b_6, \end{gathered} \tag{35}$$

where i can take on the values 3, 4, 5 or 6.

The generalized characteristics for incident p- and s-waves are the solutions of eqns (28) written as

$$z_i=\begin{cases} \dfrac{(x-x_0)t-\mathrm{i}|y-y_0|\sqrt{t^2-a_i^2r^2}}{r^2} & a_i^2r^2<t^2, \\ \dfrac{(x-x_0)a_i}{r} & a_i^2r^2=t^2, \quad i=1,2, \\ \dfrac{(x-x_0)t-|y-y_0|\sqrt{a_i^2r^2-t^2}}{r^2}. & a_i^2r^2>t^2, \end{cases} \tag{36}$$

where $r^2=(x-x_0)^2+(y-y_0)^2$.

The straight line which corresponds to $z_i=$ constant intersects the boundary plane $y=0$ in x, y, t-space at the point $(\bar{x},0,\bar{t})$ which is defined next. The boundary value of the characteristics is

$$\lambda_i=z_i(\bar{x},0,\bar{t})=\begin{cases} \dfrac{(\bar{x}-x_0)\bar{t}-\mathrm{i}y_0\sqrt{\bar{t}^2-a_i^2\bar{r}^2}}{\bar{r}^2} & a_i^2\bar{r}^2<\bar{t}^2 \\ \dfrac{(\bar{x}-x_0)a_i}{\bar{r}} & a_i^2\bar{r}^2=\bar{t}^2 \end{cases} \quad i=1,2, \tag{37}$$

where $\bar{r}^2=(\bar{x}-x_0)^2+y_0^2$.

The reflected characteristics z_3 and z_4 in eqns (35) pass through the point $(\bar{x},0,\bar{t})$, and z_1, z_3 and z_4 all satisfy

$$\bar{t}-\bar{x}z_i+x_0z_i-\beta_i(z_i)y_0=0, \quad i=1,3,4. \tag{38}$$

That is, $z_3(\bar{x},0,\bar{t})=z_4(\bar{x},0,\bar{t})=\lambda_1$. Substituting λ_1 into eqns (35) yields straight lines for the generalized characteristics of pp- and ps-waves whose equations are

$$t-\bar{t}-(x-\bar{x})\lambda_1-y\eta_i(\lambda_1)=0, \quad i=3,4. \tag{39}$$

As $\bar{t}=a_1\bar{r}$, $\lambda_1=(\bar{x}-x_0)a_1/\bar{r}$, eqns (39) are the equations for the wavefronts of pp- and ps-waves (see Fig. 1):

$$\begin{gathered} f_3(x,y,t)=t-a_1\sqrt{(x-x_0)^2+(y+y_0)^2}=0, \\ f_4(x,y,t)=t-a_1\sqrt{(\bar{x}-x_0)^2+y_2^2}-a_2\sqrt{(x-\bar{x})^2+y^2}=0, \end{gathered} \tag{40}$$

where $\bar{x}$ is the solution of the following equation from $z_4(\bar{x},0,\bar{t})=\lambda_1$:

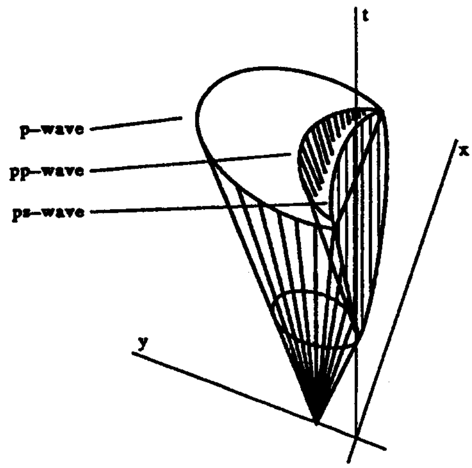

Fig. 1. Wavefronts of incident p-wave and reflected waves in xyt-space.

$$\frac{(\bar{x}-x_0)a_1}{\sqrt{(\bar{x}-x_0)^2+y_0^2}}=\frac{(x-\bar{x})a_2}{\sqrt{(x-\bar{x})^2+y^2}}. \tag{41}$$

Note that eqn (41) is Snell's law.

Similarly for the incident s-wave the wavefront of the sp- and ss-waves can be obtained by putting $\bar{t}=a_2\bar{r}$ and $\lambda_2=(\bar{x}-x_0)a_2/\bar{r}$ (see Fig. 2) leading to

$$\begin{aligned} f_5(x,y,t) &= t-a_2\sqrt{(\bar{x}-x_0)^2+y_0^2}-a_1\sqrt{(x-\bar{x})^2+y^2}=0, \\ f_6(x,y,t) &= t-a_2\sqrt{(x-x_0)^2+(y+y_0)^2}=0, \end{aligned} \tag{42}$$

where $\bar{x}$ is the solution of the following equation:

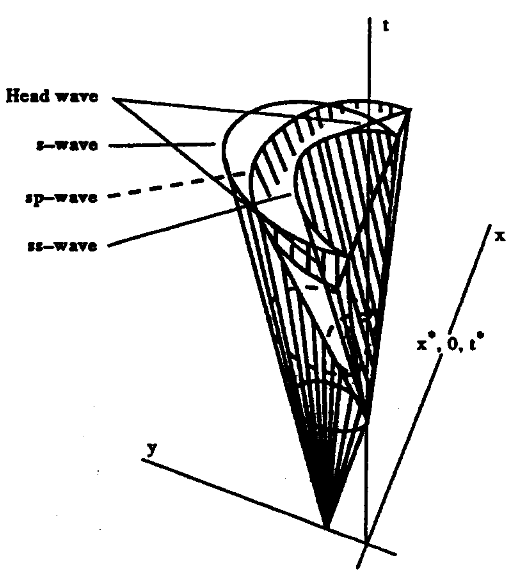

Fig. 2. Wavefronts of incident s-wave and reflected waves in xyt-space.

$$\frac{(\bar{x}-x_0)a_2}{\sqrt{(\bar{x}-x_0)^2+y_0^2}}=\frac{(x-\bar{x})a_1}{\sqrt{(x-\bar{x})^2+y^2}} \quad \text{and} \quad \bar{x}<x^*, \quad x^* \text{ see below.} \tag{43}$$

Now the head wave is analysed. The wavefront of the incident s-wave at $y = 0$ is expressed as eqn (7) resulting in

$$\bar{t} = a_2\sqrt{(\bar{x}-x_0)^2+y_0^2} \quad \text{as} \quad y = 0. \tag{44}$$

The rate of s-wave travel along the traction free surface is

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}}=\frac{\bar{r}}{(\bar{x}-x_0)a_2}. \tag{45}$$

When

$$\frac{1}{a_1}>\frac{d\bar{x}}{\mathrm{d}\bar{t}}, \quad \text{or} \quad \bar{t}>a_2y_0\sqrt{a_2^2/(a_2^2-a_1^2)}, \tag{46}$$

the reflected sp-wave propagates along the traction-free boundary faster than the incident wave, which leads to head waves. The critical point of generating head waves is at $(x^*, 0, t^*)$ where

$$x^* = x_0 \pm y_0\sqrt{a_1^2/(a_2^2-a_1^2)}, \quad t^* = a_2y_0\sqrt{a_2^2/(a_2^2-a_1^2)}. \tag{47}$$

After t^* (or $\bar{x} > x^*$), that portion of the wavefront of the sp-wave can be determined by

$$t-t^*-(x-x^*)z_5-y\sqrt{a_1^2-z_5^2}=0$$

or

$$t-t^* = a_1\sqrt{(x-x^*)^2+y^2}. \tag{48}$$

The characteristic at $y = 0$ is

$$\lambda^* = z_5(x,0,t) = \frac{t-t^*}{x-x^*} = \pm a_1. \tag{49}$$

The wavefront of the head wave depends on λ^* as

$$t\pm(x-x_0)a_1-(y+y_0)\sqrt{a_2^2-a_1^2} = 0. \tag{50}$$

Equations (47) and (50) along with the front of the reflected ss-wave [the second of eqn (42)] construct the region of influence of the ss head wave. In this region, z_6 is expressed using eqns (35) as

$$z_6 = \frac{t(x-x_0)\mp|y-y_0|\sqrt{a_1^2r^2-t^2}}{r^2}. \tag{51}$$

For $t < a_ir$ in eqns (36), the incident wave is a plane wave. The reflection of such plane waves can be discussed in a similar manner. If $y = 0$ or $y_0 = 0$, eqn (50) and the second of eqns (42) are the same as the regions of influence of the head waves found for these special cases by Payton (1967) and Norwood (1973).

Conversely, if x, y, t in eqns (35) are given, $z_i(x, y, t)$ $(i = 3, 4, 5, 6)$ can be found after some effort that involves finding the solutions to quartic equations. Standard formulae for solving the pertinent quartic equation can be used (Spiegel, 1968).

The disturbances caused by reflected waves can be written from eqns (21) and (24) as

$$G_{ij}(z_i) = K_i^u(z_i)G_{mj}(z_i), \quad H_{ij}(z_i) = K_i^v(z_i)H_{mj}(z_i),$$
$$m = 1,2, \quad i = 2m+n, \quad n = 1,2, \quad j = 1,2, \tag{52}$$

where $K_3 = K_{pp}$, $K_4 = K_{ps}$, $K_5 = K_{sp}$, and $K_6 = K_{ss}$ are given in eqns (22) and (25). Finally the solution of Lamb's problem for a buried force is given as

$$u_j(x,y,t) = \mathrm{Re}\sum_{k=1}^{6} G_{kj}(z_k)S_k, \quad v_j(x,y,t) = \mathrm{Re}\sum_{k=1}^{6} H_{kj}(z_k)S_k, \quad j = 1,2, \tag{53}$$

where

$$S_i = [(x-x_0) - yz_i/\eta_i(z_i) - y_0 z_i/\beta_i(z_i)]^{-1}. \tag{54}$$

Similar to eqns (34), the denominators of S_k in eqns (53) can be zero giving singular solutions corresponding to the wavefronts.

The regions of influence of the reflected waves and the head waves are given automatically by the displacement expressions (34) or (53). For example:

(1) For the *sp*-wave, the horizontal displacement due to the vertical impulse is given by

$$u_2^{sp} = \begin{cases} \dfrac{I_y}{2\pi}\,\mathrm{Im}\left[K_{sp}^u(z_5)\dfrac{z_5\eta_5(z_5)}{(x-x_0)\eta_5(z_5) - yz_5 - y_0\eta_5(z_5)/\beta_5(z_5)}\right], & z_5^2 > a_1^2 \text{ or } z_5 \text{ is complex}, \\ 0, & z_5^2 < a_1^2 \text{ and } z_5 \text{ is real}. \end{cases} \tag{55}$$

The equation for the *sp*-wavefront is the same as that given in eqn (42).

(2) For the *ss*-wave, the horizontal displacement is given by

$$u_2^{ss}(x,y,t) = \begin{cases} \dfrac{I_y}{2\pi\sqrt{t^2-a_2^2r^2}}\,\mathrm{Re}\,[K_{ss}^u(z_6)z_6\eta_6(z_6)], & z_6^2 > a_2^2 \text{ or } z_6 \text{ is complex}, \\ \dfrac{I_y}{2\pi\sqrt{a_2^2r^2-t^2}}\,\mathrm{Im}\,[K_{ss}^u(z_6)z_6\eta_6(z_6)], & a_1^2 < z_6^2 < a_2^2 \text{ and } z_6 \text{ is real}, \\ 0, & z_6^2 < a_1^2 \text{ and } z_6 \text{ is real}, \end{cases} \tag{56}$$

where the following expressions are used:

$$(x-x_0)\eta_6(z_6) - (y+y_0)z_6 = \begin{cases} \mathrm{i}\sqrt{t^2-a_2^2r^2}, & t^2 > a_2^2r^2, \\ \sqrt{a_2^2r^2-t^2}, & t^2 < a_2^2r^2. \end{cases} \tag{57}$$

The first condition in eqn (56) is identified as eqn (42), while the second condition is obtained by putting $z_6^2 = a_1^2$ into eqns (35) leading to eqn (50).

The solution expressed in eqn (53) is equivalent to the formulae of Hudson (1980), his eqns (7.66) and (7.67), which were obtained using double integral transforms.

Writing eqns (53) along $y = 0$ gives the boundary values of the Green's functions in the half plane as

$$u_j(x,0,t) = \mathrm{Re} \sum_{k=1}^{2} F_k(z_{k0})G_{kj}(z_{k0})S_{k0},$$

$$v_j(x,0,t) = \mathrm{Re} \sum_{k=1}^{2} F_{3-k}(z_{k0})H_{kj}(z_{k0})S_{k0}, \quad j = 1,2, \tag{58}$$

where

$$F_1(z) = 4a_2^2\eta_1(z)\eta_2(z)/R(z), \quad F_2(z) = 2a_2^2\gamma(z)/R(z),$$
$$S_{k0} = [(x-x_0)-y_0z_{k0}/\eta_k(z_{k0})]^{-1}. \tag{59}$$

Head waves and the regions of influence of the waves are included implicitly in eqns (58).

When the load is applied at the boundary of the half-plane so that $y_0 = 0$, the solutions for Lamb's problem for a surface load can be solved directly. In this case, the governing equations are homogeneous and the boundary conditions are inhomogeneous. Noting that there are no reflected waves, the generalized characteristics are written as

$$t-z_i(x-x_0)-y\eta_i(z_i) = 0, \quad i = 1,2. \tag{60}$$

Satisfying the stress free boundary conditions leads to

$$G_1^b(z_1) = [2A_1z_1^2\eta_2(z_1)+A_2z_1\gamma(z_1)]/R(z_1),$$
$$G_2^b(z_2) = [A_1\eta_2(z_2)\gamma(z_2)-2A_2z_2\eta_1(z_2)\eta_2(z_2)]/R(z_2),$$
$$H_1^b(z_1) = G_1^b(z_1) * \eta_1(z_1)/z_1, \quad A_1 = \mathrm{i}I_x/\pi,$$
$$H_2^b(z_2) = -G_2^b(z_2) * z_2/\eta_2(z_2), \quad A_2 = \mathrm{i}I_y/\pi. \tag{61}$$

The components of displacement are obtained as

$$u = a_2^2 \,\mathrm{Re} \sum_{i=1}^{2} G_i^b(z_i)S_i(z_i), \quad v = a_2^2 \,\mathrm{Re} \sum_{i=1}^{2} H_i^b(z_i)S_i(z_i), \tag{62}$$

where

$$S_i(z_i) = [(x-x_0)-yz_i/\eta_i(z_i)]^{-1}. \tag{63}$$

The wavefront can be obtained by putting $y_0 = 0$ in eqns (28). Equations (62) are identical to formulae that represent the solution to Lamb's problem for a surface load [see Craggs (1960), Miles (1960), Thompson and Robinson (1969), Cherepanov and Afanas'ev (1974) and Norwood (1973)]. In this special case, the present method is the same as Smirnov–Sobolev's method and eqns (60) are equivalent to Chaplygin's transformations.

7. NUMERICAL RESULTS

In this section, the solution is illustrated through the calculation of displacement components at several receiver locations. Both receivers near to and far away from the source and on or under the traction free surface are considered. For compactness, results are given only for the vertical applied force ($I_x = 0$). Results are given as nondimensional horizontal and vertical displacements, $u(x,y,t)\pi c_2 y_0/I$ and $v(x,y,t)\pi c_2 y_0/I$, versus nondimensional time, tc_2/r, where r is the distance between the receiver location and source point. The coordinates of the receivers are given in terms of the nondimensional distances $n = (x-x_0)/y_0$ and $m = y/y_0$. The ratio of the speeds of the p- and s-waves (a_2/a_1) is taken as $\sqrt{3}$ for all of the numerical examples. The infinite displacements occurring at some of the wavefronts are shown as the maximum in each scale.

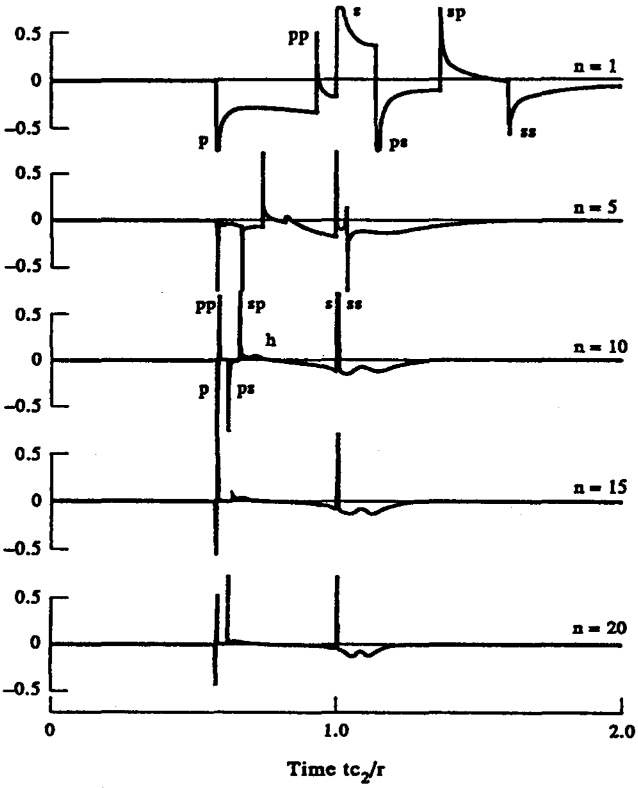

Fig. 3. Dimensionless horizontal displacement history $(u(t)\pi c_2 y_0/I_x)$ for $n = 1, 5, 10, 15, 20$ and $m = 0.5$.

Figures 3 and 4 show horizontal and vertical displacement histories of several sub-surface receivers located at $n = 5, 10, 15, 20, 25$ and $m = 0.5$. The time of arrival of each wave is clearly discernible in both the horizontal and vertical displacement histories. The displacement due to the direct p- and s-waves is the main feature for the receivers near the source point.

Figure 5 shows the vertical displacement history at various depths along the vertical line through the source point. The results emphasize that the solutions due to the direct and reflected wavefronts are not always infinite. Figure 6 shows the vertical displacement history for surface receivers both relatively near the source and quite far away from the source. The Rayleigh pulse attenuates very slowly along the x-direction. Equations (58), which cover the special case of $y = 0$, are very useful for the generation of the results in Fig. 6 as evaluation using the full solution [eqns (53)] requires great care for large values of x.

Figure 7 shows results analogous to the second graph in Fig. 3 with the displacement decomposed into the separate waves. The displacements for these separate waves approaches infinity as time approaches infinity. However, superposition of the results from each wave leads to zero displacement at large values of time.

8. CONCLUSION

The characteristic conditions of the Navier equations for elastic waves exist as both real and complex solutions in x, y, t-space. They are called generalized characteristics. The generalized characteristics have the same properties as the real characteristics. The first order differential equations along the generalized characteristics are equivalent to the governing equations in the sense that their solutions automatically satisfy the governing

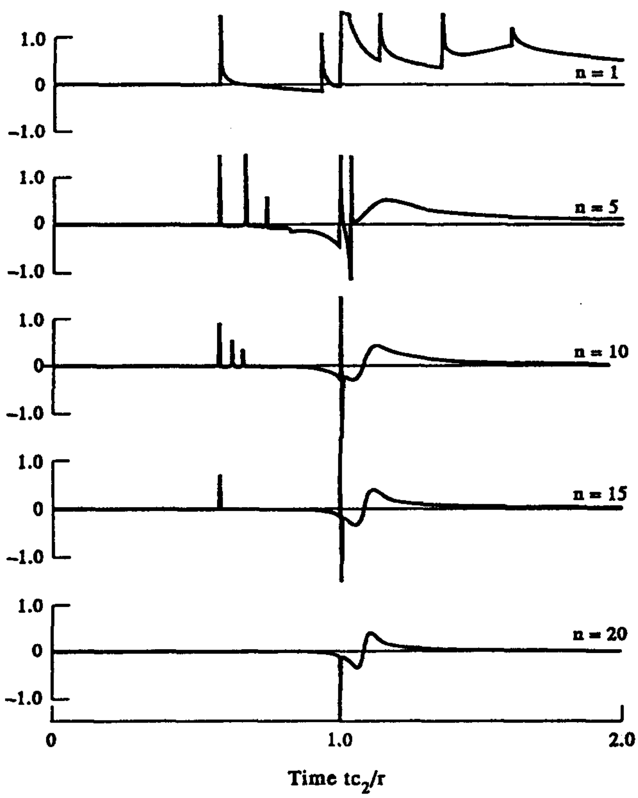

Fig. 4. Dimensionless vertical displacement history $(v(t)\pi c_2 y_0/I_x)$ for $n = 1, 5, 10, 15, 20$ and $m = 0.5$.

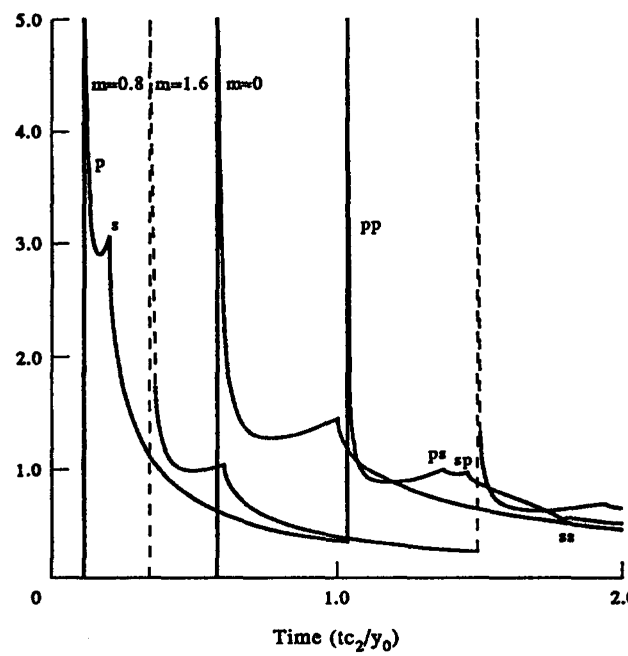

Fig. 5. Dimensionless vertical displacement history $(v(x, y, t)\pi c_2 y_0/I_y)$ for several receivers at $n = 0.0$, $m = 0.0, 0.8, 1.6$.

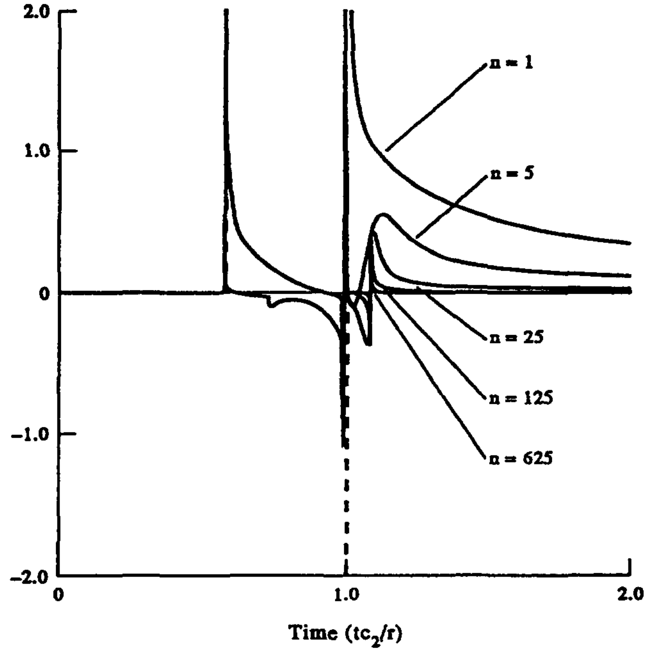

Fig. 6. Dimensionless vertical displacement history ($v(x, y, t)\pi c_2 y_0/I_y$) for surface receivers at $n = 1$, 5, 25, 125, 625, $m = 0$.

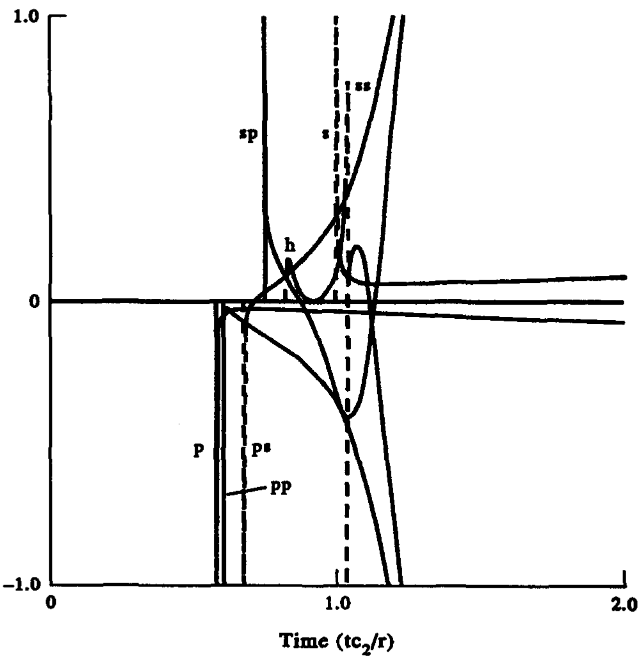

Fig. 7. Dimensionless horizontal displacement history ($u(x, y, t)\pi c_2 y_0/I_y$) for separate waves at $n = 5.0$, $m = 0.5$.

equations. Using this method, a boundary value problem of elastodynamics is solved in two steps: the first step is the calculation of generalized characteristics with geometric boundary conditions and the second is the calculation of the displacement or its derivatives using stress (or displacement) boundary conditions. Each step involves the solution of first order (partial or ordinary) differential equations. The method developed in the present paper makes performance of the inverse Laplace transform required by Cagniard's technique unnecessary.

The method also can be used to solve problems of elastodynamics for transversely isotropic media and of electromagnetic radiation. Additionally, the method may be applied to three-dimensional problems of elastodynamics and electromagnetics; however, here the details are complicated.

Acknowledgement—The research reported in this paper was supported in part by NSF under Grant MSS-9057082.

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APPENDIX A

If the characteristic surface is given in the form

$$t = \psi_i(x, y, C_i) \tag{64}$$

then the characteristic condition can be simplified to

$$\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2-a_1^2\right]\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2-a_2^2\right]=0. \tag{65}$$

Following Charpit's method as described in Chapter 2 of Sneddon (1957), the general solution to first order nonlinear partial differential equations of the form given by eqn (65) can be solved using separation of variables as follows. Assuming $\psi_i(x, y) = \xi_i(x)+\eta_i(y)$, eqn (65) leads to

$$(\xi_i')^2+(\eta_i')^2 = a_i^2 \quad \text{or} \quad (\xi_i')^2 = a_i^2-(\eta_i')^2, \quad i = 1, 2. \tag{66}$$

Since the right-hand side of eqn (66) is independent of x and its left-hand side is independent of y, each side of the equation is equal to the same constant C_i^2. This results leads to solutions [see Chapter I and Chapter II of Courant and Hilbert (1962)]:

$$\psi_i = C_i x \mp \sqrt{a_i^2-C_i^2}\,y-\phi_i(C_i), \quad i = 1, 2, \tag{67}$$

where C_i is an arbitrary parameter and $\phi_i(C_i)$ is an arbitrary function. Thus, the characteristic surface is given by

$$t-C_i x \pm \sqrt{a_i^2-C_i^2}\,y+\phi_i(C_i) = 0, \quad i = 1, 2. \tag{68}$$

APPENDIX B

For example, if $\phi_i(z_i) \equiv 0$ and $x > 0$ in eqn (10), the solution of eqn (10) is

$$z_i = \frac{xt \pm \mathrm{i}|y|\sqrt{t^2-a_i^2r^2}}{r^2} \tag{69}$$

and the function η_i can be expressed as

$$\eta_i = \pm\frac{|y|t \mp \mathrm{i}x\sqrt{t^2-a_i^2r^2}}{r^2}. \tag{70}$$

Substituting z_i and η_i eqn (10) leads directly to the result that $\mathrm{Re}(\eta_i) \geqslant 0$ and $\mathrm{Im}(\eta_i) < 0$ if $\mathrm{Im}(z_i) > 0$ or $\mathrm{Im}(\eta_i) > 0$ if $\mathrm{Im}(z_i) < 0$.

For the general case of $\phi_i(z_i)$ not identically equal to zero, similar results are obtained by using eqn (10).

For completeness it is noted that either choice of sign in eqn (69) leads to the same solution of eqn (1).

APPENDIX C

There are several definitions of the δ function [see Chapter VI of Courant and Hilbert (1962)]. One useful form is

$$\delta(\alpha) = \frac{1}{\pi}\lim_{\beta\to 0}\frac{\beta}{\alpha^2+\beta^2}.$$

This leads to

$$k\delta(\alpha) = \frac{k}{\pi}\lim_{\beta\to 0}\frac{\beta}{\alpha^2+\beta^2} = \frac{1}{\pi}\lim_{\beta\to 0}\frac{\beta/k}{(\alpha/k)^2+(\beta/k)^2} = \delta\left(\frac{\alpha}{k}\right).$$

The solution to eqn (28) is written as

$$z_i = \frac{(x-x_0)t \mp \mathrm{i}|y-y_0|\sqrt{t^2-a_i^2r^2}}{r^2}, \quad r^2 = (x-x_0)^2+(y-y_0)^2.$$

Now in the sense of generalized functions

$$\lim_{|y-y_0|\to 0} z_i = \frac{t}{x-x_0} \mp \mathrm{i}\sqrt{t^2-a_i^2(x-x_0)^2} \lim_{|y-y_0|\to 0} \frac{|y-y_0|}{r^2}$$

$$= \frac{t}{x-x_0} \mp \mathrm{i}\pi\sqrt{t^2-a_i^2(x-x_0)^2}\,\delta(x-x_0)$$

$$= \frac{t}{x-x_0} \mp \mathrm{i}\pi t\delta(x-x_0) \equiv \frac{t}{x-x_0} \mp \mathrm{i}\pi\delta\left(\frac{x-x_0}{t}\right),$$

so that

$$\delta\left(\frac{x-x_0}{t}\right) = \pm \mathrm{Re}\left\{\frac{\mathrm{i}}{\pi} \lim_{|y-y_0|\to 0} z_i\right\}.$$